18-feb Lecture 7

Wednesday, February 18, 2026 8:31 AM

- graded HW1's are published
- end of exam material is NEXT week
- correction to Lecture 6 notes

ex. revisit crossbow bolt shot straight upward with $v_0 = 49$ m/s

this time considering air resistance s.t. $\rho = 0.04 \rightarrow \frac{1}{25}$

find max height:

use prev. eqn $\frac{dv}{dt} = -\rho v - g$

$$v(t) = \left(v_0 + \frac{g}{\rho}\right)e^{-\rho t} - \frac{g}{\rho}$$

$\frac{g}{\rho} = \frac{9.8}{.04} = 245$

$$\boxed{v(t) = (\cancel{49+245})^{294} e^{-t/25} - 245}$$

$$\therefore y(t) = -25(294)e^{-t/25} - 245t + C$$ where C is arb. constant

$$= -7350e^{-t/25} - 245t + C$$

when $y(0) = 0$: $0 = -7350(1) - 0 + C$

$$C = 7350$$

$$\boxed{y(t) = -7350e^{-t/25} - 245t + 7350}$$

max height occurs when $v = 0$: $v(t) = 294e^{-t/25} - 245 = 0$

$$e^{-t/25} = \frac{245}{294}$$

$$-t/25 = \ln\left(\frac{245}{294}\right)$$

$$t = -25\ln\left(\frac{245}{294}\right)$$

$$\approx 4.558 \text{ sec.}$$

then y_{max} $y(4.558) \approx 108.28$ m

versus

122.5 m ← sol'n without air resistance

when does the bolt return to the ground?

when $y(t) = -7350e^{-t/25} - 245t + 7350 = 0$

use Newton's Method

$$t_{n+1} = t_n - \frac{y(t_n)}{y'(t_n)}$$

guess $t_0 = 10$

↑ based on max height @ 4.6 s

$$t_1 = t_0 - \frac{y(t_0)}{y'(t_0)}$$

$$= 10 - \frac{y(10)}{y'(10)}$$

$$= 10 - \frac{-26.85}{-47.83}$$

$t_1 \approx 9.4386$

$t_2 = 9.41106$

$t_3 = 9.41095$

$t_4 = 9.4114$

estimate that bolt is in air for $\boxed{9.41\text{ s}}$

speed at which bolt hits the ground $|v(9.41)| = \left|294e^{-\frac{9.41}{25}} - 245\right|$

$v \approx 43.22$ m/s

recall v_0 was 49 m/s

recall "simplistic" version $F_R = \pm kv^2$

$k > 0$

choice of sign depends on direction of motion, which F_R always opposes

y ⊕

↑ define positive y direction as upward

y ⊕ which F_R always opposes

define positive y direction as upward

then $F_R < 0$ when $v > 0$ ($\frac{dy}{dt}$)

is upward

then $F_R > 0$, $v < 0$

is downward

since sign of F_R is always opposite of sign of v

then $F_R = \pm kv^2$ CBW as $F_R = -kv|v|$

$- = - + + +$

$+ = - + - +$

$F = ma = m\frac{dv}{dt}$ and $F = F_G + F_R$

$= -mg - kv|v|$

$$m\frac{dv}{dt} = -mg - kv|v|$$

$$\frac{dv}{dt} = -g - \frac{k}{m}v|v| \quad \text{recall } \rho = \frac{k}{m} > 0$$

$$\frac{dv}{dt} = -g - \rho v|v|$$

will discuss upward motion separate from downward motion

<u>Upward Motion</u>

ex. suppose projectile is launched straight up from initial position y_0 with initial velocity $v_0 > 0$

then given $v > 0$: $\frac{dv}{dt} = -g - \rho v^2$

$$\frac{dv}{dt} = -g\left(1 + \frac{\rho}{g}v^2\right) \quad \text{Separable eqn}$$

$$\int \frac{1}{1+\frac{\rho}{g}v^2}\, dv = -g\int dt$$

let $u = \sqrt{\frac{\rho}{g}}\,v$ $\qquad \sqrt{\frac{g}{\rho}}\int \frac{1}{1+u^2}\, du$

$\sqrt{\frac{g}{\rho}}\, du = dv$ $\qquad \sqrt{\frac{g}{\rho}}\arctan u = -g\int dt \rightarrow \sqrt{g}$

$$\sqrt{\frac{g}{\rho}}\,du = dv \qquad \sqrt{\frac{g}{\rho}}\ \arctan u = -g\int dt$$

$$u^2 = \frac{\rho}{g}v^2 \qquad \arctan u = -\frac{g}{\sqrt{g}}\sqrt{\rho}\int dt \quad \sqrt{g}$$

$$\arctan\left(\sqrt{\frac{\rho}{g}}\,v\right) = -\sqrt{g\rho}\ t + C \quad \text{where } C \text{ is arb constant}$$

$$\sqrt{\frac{\rho}{g}}\,v = \tan\left(-\sqrt{g\rho}\,t + C\right)$$

$$v(t) = \sqrt{\frac{g}{\rho}}\tan\left(-\sqrt{g\rho}\,t + C\right)$$

$$\int \tan u\,du = \int \frac{1}{\cos u}\sin u\,du = -\ln|\cos u| + C \qquad \xleftarrow[\text{leave } C \text{ in } v(t)]{\text{integrate}} \qquad v(t) = \sqrt{\frac{g}{\rho}}\tan\left(C - \sqrt{g\rho}\,t\right)$$

when $v(0) = v_0$: $\quad v_0 = \sqrt{\frac{g}{\rho}}\tan(C - 0)$

$$C = \arctan\left(\sqrt{\frac{\rho}{g}}\,v_0\right)$$

then $v(t) = \sqrt{\frac{g}{\rho}}\tan\left(\arctan\sqrt{\frac{\rho}{g}}\,v_0 - \sqrt{g\rho}\,t\right)$

$\vdots$

$$\boxed{y(t) = \frac{1}{\rho}\ln\left|\frac{\cos\left(C - \sqrt{g\rho}\,t\right)}{\cos C}\right| + y_0}$$

Downward Motion

ex. suppose projectile is launched/dropped straight downward from initial position y_0 with initial velocity $v_0 \leq 0$

then given $v < 0$: $\quad \frac{dv}{dt} = -g + \rho v^2$

$$\frac{dv}{dt} = -g\left(1 - \frac{\rho v^2}{g}\right)$$

again, using $u = \sqrt{\frac{\rho}{g}}\,v$ get $\int \frac{1}{1-u^2}\,du = \tanh^{-1} u + C$

$$\therefore v(t) = \sqrt{\frac{g}{\rho}} \tanh\left(C_2 - \sqrt{\rho g}\, t\right)$$

C_2 works out to be $\tanh^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right)$

it follows from previous example that $\int \tanh u \, du = \ln|\cosh u| + C$

$$\therefore y(t) = y_0 - \frac{1}{\rho} \ln\left|\frac{\cosh\left(C_2 - \sqrt{\rho g}\, t\right)}{\cosh C_2}\right|$$

given $v_0 = 0$: $C_2 = \tanh^{-1}(0) = 0$

$$v(t) = \sqrt{\frac{g}{\rho}} \tanh\left(-\sqrt{\rho g}\, t\right)$$

$\tan(-x) = -\tan x$ odd f

$\therefore \tanh(-x) = -\tanh x$

$$v(t) = -\sqrt{\frac{g}{\rho}} \tanh\left(\sqrt{\rho g}\, t\right)$$

$\tanh x = \frac{\sinh x}{\cosh x}$

$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{\sinh x}{\cosh x} = \lim_{x\to\infty} \frac{\cancel{\frac{1}{2}}(e^{x} - e^{-x})}{\cancel{\frac{1}{2}}(e^{x} + e^{-x})}$$

$$= 1$$

"$\frac{\infty}{\infty}$"

$\lim_{x\to\infty} e^{x} = \infty$

$\lim_{x\to\infty} e^{-x} = 0$

$$v(t) = -\sqrt{\frac{g}{\rho}} \Rightarrow |v_t| = \sqrt{\frac{g}{\rho}} \text{ for downward motion}$$

<u>Variable Gravitational Acceleration</u>

gravitational force of attraction between 2 masses, M and m separated by distance r is denoted: $F = \frac{GMm}{r^2}$

where $G \approx 6.6726 \times 10^{-11}$

ex. A lunar lander is free falling towards the moon.

At an altitude of 53 km above lunar surface, its downward velocity ex. A lunar lander is free falling towards the moon.

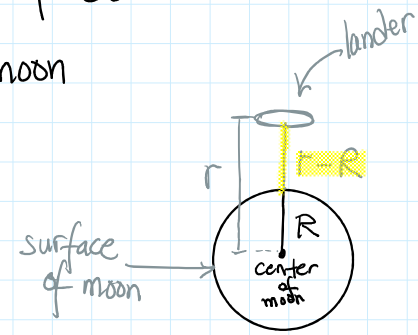
At an altitude of 53 km above lunar surface, its downward velocity is 1477 km/h

Its retrorockets provide a deceleration $T = 4\ m/s^2$

At what height above lunar surface should retrorockets be activated to ensure a "soft touchdown"?

means $v = 0$ on impact

let $r(t)$ = lander's distance from $\underline{\text{center}}$ of moon



$$F = \frac{GMm}{r^2} \Rightarrow \frac{F}{m} = \frac{GM}{r^2}$$

lunar acceleration

thrust acceleration

then [net] acceleration $\frac{d^2r}{dt^2} = T - \frac{GM}{r^2}$

no variable t

M = mass moon $= 7.35 \times 10^{22}$ kg

R = radius moon $= 1.74 \times 10^{6}$ m (1740 km)

let $v = \frac{dr}{dt}$

$$\frac{dv}{dt} = \frac{d^2r}{dt^2} = \frac{dv}{dr}\cdot\frac{dr}{dt} = \frac{dv}{dr}\cdot v$$

[will finish next week]